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Transfers, Contracts and Strategic Games*

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Abstract

This paper analyses the role of transfer payments and strategic contracting within two-person strategic form games with monetary payoffs. First, it introduces the notion of transfer equilibrium as a strategy combination for which individual stability can be supported by allowing the possibility of transfers of the induced payoffs. Clearly, Nash equilibria are transfer equilibria, but under common regularity conditions the reverse is also true. This result typically does not hold for finite games without the possibility of randomisation, and transfer equilibria for this particular class are studied in some detail.

The second part of the paper introduces, also within the setting of finite games, contracting on monetary transfers as an explicit strategic option, resulting in an associated two-stage contract game. In the first stage of the contract game each player has the option of proposing transfer schemes for an arbitrary collection of outcomes. Only if the players fully agree on the entire set of transfer proposals, the payoffs of the game to be played in the second stage are modified accordingly. The main results provide explicit characterisations of the sets of payoff vectors that are supported by Nash equilibrium and virtual subgame perfect equilibrium, respectively.

Keywords: monetary transfer scheme, transfer equilibrium, contract game, virtual subgame perfect equilibrium, Folk theorems

JEL classification: C72

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1 Introduction

This paper investigates the role of allowing certain aspects of commitment and cooperation within the framework of strategic form games. More in particular, it focuses on the explicit strategic option of costless contracting on monetary transfer schemes with respect to particular outcomes.

Closely related to the current paper are the papers of Jackson and Wilkie (2005) and Yamada (2003). Basically both of these papers allow for a rather broad type of contracts within a setting of mixed extensions of finite strategic form games. In this setting contracts on transfer payments are contingent on the actual choice of specific strategies. Jackson and Wilkie (2005) illustrate that this type of costless contracting does not necessarily lead to efficiency. Yamada (2003) explicitly models the described format of contracting as a strategic option in a two-stage extensive form game and derives a kind a Folk theorem: the payoff configurations supported by subgame perfect Nash equilibria of this two-stage contract game are characterised.

The objective of the current paper is to analyse contracts on transfer payments of only the simplest form. The contracting will be contingent on the actual occurrence of outcomes, so only on the realisation of strategy combinations and not on the actual choice of individual strategies. This lowers the degree of sophistication required in the cooperative commitments. In particular, it avoids intrinsic problems regarding the non-perceptibility of mixed strategies. Moreover, the two-stage contract game in the current paper allows only for a unanimity type of contracting on sets of outcomes. By restricting to this type of basic contracting and combining this with the more appropriate concept of *virtual* subgame perfection as introduced by García-Jurado and Gonzalez-Díaz (2006), Yamada's Folk theorem is recovered.

Although the concepts and results in this paper can be readily extended to games with more players, we restrict our attention to two-player games for expositional purposes.

The first part of the paper deals with the possibility of making a specific strategy combination individually stable by having a simple monetary transfer scheme contingent on the actual realisation of the corresponding outcome.

Such a strategy combination is called a transfer equilibrium. Under standard regularity conditions however (which for example are satisfied for any mixed extension of a finite game) it turns out that the set of transfer equilibria coincides with the set of Nash equilibria. For finite games without the possibility of randomisation, the set of Nash equilibria can be a strict subset of the set of transfer equilibria. This particular subclass is analysed in some detail.

The second and larger part of the paper models contracting on monetary transfers as an explicit strategic option within a two-stage extensive form setting. The first stage consists of the contracting stage where both players can propose transfer schemes as before but now possibly on multiple outcomes simultaneously. Only if both players fully agree on all transfer proposals ("give or take"), the payoffs of the original game are modified accordingly and the modified game is played in the second stage. So both the type of contract proposals and the subsequent implementation mechanism of the proposals are as simple as possible. It is important to note that in this setting implemented contracts on transfer schemes with respect to certain outcomes may lead to the rise of equilibria at outcomes that are not specified in the contracts.

The first main result is a full characterisation of all equilibrium payoff vectors in the same spirit as the well-known Folk theorems in the context of repeated games. It turns out that exactly those payoff vectors that are bounded from below by the individual minimax payoffs and for which the total sum of the payoffs is bounded from above by the maximum of the total payoffs over all outcomes, correspond to Nash equilibria of the two-stage contract game. After arguing that the set of subgame perfect Nash equilibria (cf. Selten (1965)) of the contract game is empty because of the non-existence of equilibria in seemingly irrelevant subgames, we focus attention on the notion of virtual subgame perfect equilibrium (cf. García-Jurado and Gonzalez-Díaz (2006)). This notion seems especially relevant and suitable in our framework. Roughly speaking, virtual subgame perfection requires players to play best responses only in subgames close to the equilibrium path. The second main result states that exactly those payoff vectors that are individually bounded from below by some equilibrium payoff, and for which there is a similar upper bound as in the case of Nash equilibria, correspond to virtual subgame perfect equilibria of the contract game.

The outline of this paper is as follows. Section 2 analyses the possibility of contracting on monetary transfer with respect to one particular outcome and investigates the corresponding notion of transfer equilibrium. In Section 3 the two-stage contract game that allows for strategic contracting on sets of outcomes is formally introduced and explained. Section 4 states and proves the Folk-like theorems with respect to Nash equilibria and virtual subgame perfect equilibria of the contract game.

2 Transfer equilibria

A *two-player strategic game* is a quartet $G = (X_1, X_2, H_1, H_2)$, where X_i denotes the strategy set of player $i \in \{1, 2\}$ and $H_i : X \to \mathbb{R}$ is his payoff function, assigning to each strategy profile $x = (x_1, x_2) \in X$ (with $X = X_1 \times X_2$) a payoff $H_i(x)$. In our framework we allow for certain transfers of payoff from one player to the other, so we assume the payoffs to be monetary.

A *Nash equilibrium* of G is a strategy profile $x \in X$ such that $H_i(x) \geq H_i(x'_i, x_{-i})$ for all $x'_i \in X_i$ and $i \in \{1, 2\}$. A Nash equilibrium is usually predicted as the outcome of a game when players are not able to make binding agreements on their strategy choices, but they are allowed to communicate before play starts.

In this paper, we allow the players to cooperate in a limited way. We assume that they have a mechanism which allows them to make an enforceable commitment before play starts on a transfer of money after both players have chosen their pre-specified strategy. So, players can agree to commit themselves to any reallocation of $H_1(x) + H_2(x)$, conditional on the outcome $x \in X$.

Both players also have the option not to cooperate in this way.[5] So, we have to make a distinction between the two possible partitions of the player set. This collection of partitions is denoted by $\mathcal{P} = \{\{\{1\}, \{2\}\}, \{\{1, 2\}\}\}$.

[5] Note that if there are more players, we also should allow for partial cooperation, which naturally leads to a partition of the player set into cooperating components.

Definition 2.1 *A* transfer equilibrium *of G is a strategy combination $x \in X$ for which there exist $(y, P) \in \mathbb{R}^2 \times \mathcal{P}$ such that*

$$i) \quad \sum_{i \in S} y_i = \sum_{i \in S} H_i(x) \qquad \textit{for all } S \in P, \tag{2.1}$$

$$ii) \quad y_i \geq H_i(x_i', x_{-i}) \qquad \textit{for all } i \in N \textit{ and all } x_i' \in X_i \setminus \{x_i\}. \tag{2.2}$$

The concept of transfer equilibrium is a generalisation of the concept of Nash equilibrium, as is stated in the following lemma.

Lemma 2.2 *Each Nash equilibrium is a transfer equilibrium.*

Note that the reverse of this lemma also holds for any transfer equilibrium x that is supported by $(y, P) = (H(x), \{\{1\}, \{2\}\})$.

The following proposition shows that if G satisfies some regularity conditions, all transfer equilibria correspond to Nash equilibria.

Proposition 2.3 *Let X_1 and X_2 be convex subsets of finite-dimensional Euclidean spaces and let H_1 and H_2 be continuous. Then x is a Nash equilibrium of G if and only if x is a transfer equilibrium of G.*

Proof: In view of Lemma 2.2, we only have to show the "if" part. Suppose that x is a transfer equilibrium and assume that both X_1 and X_2 have more than one element (otherwise the proof is more straightforward). Let $i \in \{1, 2\}$ and let $\varepsilon > 0$. Let $x_i' \in X_i \setminus \{x_i\}$ be such that $|H_i(x) - H_i(x_i', x_{-i})| < \varepsilon$. Note that such an x_i' always exists because X_i is a convex subset of an Euclidean space and H_i is continuous. Then (2.2) implies that $y_i \geq H_i(x) - \varepsilon$. Since this holds for every $\varepsilon > 0$, (2.1) implies that $y_i = H_i(x)$ for all $i \in \{1, 2\}$. Thus (2.2) implies that x is a Nash equilibrium of G. $\square$

Given Proposition 2.3, we restrict our attention to games that do not satisfy the regularity conditions mentioned there. In particular, we consider games with a finite number of strategies. A *finite two-player game* is a quartet $G = (M, N, H_1, H_2)$, where $M = \{1, \dots, m\}$ and $N = \{1, \dots, n\}$ are the strategy sets of player 1 and 2, respectively. G is usually denoted by a pair of payoff matrices (A, B), where $A = (a_{ij} = H_1(i,j))_{(i,j)\in M\times N}$ and $B = (b_{ij} = H_2(i,j))_{(i,j)\in M\times N}$.

The following examples illustrate the concept of transfer equilibria for finite two-player games. The first example is a prisoners' dilemma and shows that in such a game, the set of transfer equilibria may contain elements that are not Nash equilibria.

Example 2.1 Consider the following finite two-player game (A, B):

$$\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \left[\begin{array}{cc} 3,3 & 0,5 \\ 5,0 & 1,1 \end{array}\right] \end{array}$$

Using Lemma 2.2, it is immediately clear that (B, R) is a transfer equilibrium (with $y = H(x)$ and $P = \{\{1\},\{2\}\}$). Furthermore, (B, R) is also supported as a transfer equilibrium by $\{((x, 2-x), \{\{1,2\}\}) \mid x \in [0,2]\}$. The other transfer equilibria are (B, L), with $\{((x, 5-x), \{\{1,2\}\}) \mid x \in [3,4]\}$, and the mirror image (T, R) with $\{((x, 5-x), \{\{1,2\}\}) \mid x \in [1,2]\}$.

Note that the concept of transfer equilibria is different from full cooperation, as in that case the players would play (T, L) and divide a total amount of 6 between them. This is, however, impossible as for any transfer of money in that cell, at least one player has an incentive to deviate. $\triangleleft$

In the approach of Jackson and Wilkie (2005), the game in Example 2.1 does not have an equilibrium (not even (B, R)). Because of the way they set up their transfer proposals, mixing has to be allowed to sustain any equilibrium in this particular game.

The next example, known as matching pennies, demonstrates that the set of transfer equilibria can be non-empty even when there are no Nash equilibria.

Example 2.2 Consider the following finite two-player game (A, B):

$$\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \left[\begin{array}{cc} 2,0 & 0,2 \\ 0,2 & 2,0 \end{array}\right] \end{array}$$

It is obvious that the set of Nash equilibria of this game is empty. However, (T, L) supported by $((0, 2), \{\{1, 2\}\})$ is a transfer equilibrium. In fact any combination of strategies gives rise to a transfer equilibrium in an analogous way. $\triangleleft$

Although the set of transfer equilibria is an extension of the set of Nash equilibria, not all games have transfer equilibria.

Example 2.3 Consider the following finite two-player game (A, B):

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ M \\ B \end{array} & \begin{bmatrix} 6,6 & 1,9 \\ 4,7 & 6,6 \\ 4,5 & 6,4 \end{bmatrix} \end{array}$$

For none of the strategy combinations it is possible to transfer money in such a way that both players have no incentive to deviate. Therefore, this game has no transfer equilibria. $\triangleleft$

Non-existence of a transfer equilibrium in Example 2.3 follows from the following proposition, which provides a necessary and sufficient condition for a transfer equilibrium in a finite two-player game to exist.

Proposition 2.4 *Let (A, B) be a finite two-player game. Strategy profile $(r, s) \in M \times N$ is a transfer equilibrium of (A, B) if and only if*

$$a_{rs} + b_{rs} \geq a^r_{Is} + b^s_{rJ}, \tag{2.3}$$

where $a^r_{Is} = \max_{i \in M \setminus \{r\}} a_{is}$ and $b^s_{rJ} = \max_{j \in N \setminus \{s\}} b_{rj}$.

Proof: "If". Let $(r, s) \in M \times N$ be such that (2.3) holds. If both $a_{rs} \geq a^r_{Is}$ and $b_{rs} \geq b^s_{rJ}$ hold, then (r, s) is a Nash equilibrium and the result follows from Lemma 2.2. Otherwise, assume without loss of generality that $a_{rs} > a^r_{Is}$ and $b_{rs} < b^s_{rJ}$. We construct a transfer equilibrium x supported by (y, P) by defining $x = (r, s)$, $y_1 = a^r_{Is}$, $y_2 = b_{rs} + a_{rs} - a^r_{Is}$ and $P = \{\{1, 2\}\}$. Clearly

$y_1 + y_2 = a_{rs} + b_{rs} = H_1(x) + H_2(x)$, so (2.1) holds. For (2.2), $y_1 \geq a_{is}$ for every $i \in M \setminus \{r\}$ and

$$y_2 = b_{rs} + a_{rs} - a^r_{Is} \geq a^r_{Is} + b^s_{rJ} - a^r_{Is} = b^s_{rJ} \geq b_{rj}$$

for every $j \in N \setminus \{s\}$.
"Only if". Let $(r, s) \in M \times N$, supported by $((y_1, y_2), P) \in \mathbb{R}^2 \times \mathcal{P}$ be a transfer equilibrium of (A, B). If (r, s) is a Nash equilibrium of (A, B) then $a_{rs} \geq a^r_{Is}$ and $b_{rs} \geq b^s_{rJ}$, from which the assertion follows immediately. If (r, s) is not a Nash equilibrium of (A, B), then $P = \{\{1, 2\}\}$, otherwise (2.2) fails for at least one of the players. (2.1) and (2.2) then give $y_1 + y_2 = a_{rs} + b_{rs}$ and $y_1 \geq a^r_{Is}$, $y_2 \geq b^s_{rJ}$. Hence, $a_{rs} + b_{rs} \geq a^r_{Is} + b^s_{rJ}$. □

One consequence of Proposition 2.4 is that for any game with $m \leq 2$ and $n \leq 2$, the set of transfer equilibria is non-empty. Proposition 2.4 also implies that if there exists a transfer equilibrium (r, s) of (A, B), then a_{rs} is the maximum in column s of the matrix A or b_{rs} is the maximum in row r of matrix B. So, when looking for a transfer equilibrium, only the cells containing those maxima should be considered, which means that only $m+n$ checks are needed.

3 Strategic transfer contracts

In the setup of transfer equilibria as discussed in the previous section, the players have a mechanism to enforce certain commitments between them. This mechanism can be seen as a type of contract in order to transfer money between the players that is executed in case a particular strategy profile is played. By looking at the mechanism from that perspective one could however argue that the contracting possibilities of the players are quite limited. First of all players are only allowed to sign a single contract and secondly, it is required that the combination of the contract itself and the strategy profile on which it is enforced, constitutes an equilibrium.

In order to overcome these limitations we introduce for the class of two-player finite games a different and more sophisticated contracting model in this section. We assume that before playing the game, the players know which particular allocations of earnings are available. Then each player proposes a *set* of contracts. A single contract describes for one particular strategy combination a reallocation of the corresponding payoffs. We specifically allow the players to propose contracts that discard money. Only in case both players agree on the *entire* contract proposal, the game is modified according to the contract conditions.

Definition 3.1 *Let (A, B) be a finite two-player game. We define a* transfer contract *as a pair $((r, s), (y_1, y_2)) \in M \times N \times \mathbb{R}^2$ such that $y_1 + y_2 \leq a_{rs} + b_{rs}$.*

Using these transfer contracts, the players play the two-stage game described as follows.

> *First stage.* Each player i chooses a collection of transfer contracts $\alpha_i \subset M \times N \times \mathbb{R}^2$, at most one for each strategy profile. The choices are made simultaneously and independently. After both players have made their choice, the proposed contracts are publicly announced.
>
> *Second stage.* If both players have chosen the same set of transfer contracts in the first stage, this set is adopted, the payoffs of the game are modified accordingly and the players play this modified game. If the proposed contracts in the first stage do not match, the original game (A, B) is played.

We want to point out once more that contracts only come into effect in case both sets of proposed contracts coincide completely. It is therefore not possible that only part of the proposed contracts are enforced. This seems quite natural, as the preference of a player for a set of contracts does not automatically imply that he is also interested in any subset of these contracts.

The main difference between this model and the setup in Section 2 is that here contracts are a *strategic* option, as they can be signed on every cell and are not necessarily located at an equilibrium of the ensuing second stage. In particular, it is possible that a contract on one cell results in an equilibrium at another cell.

The *contract game* as described above can be represented by an extensive form game[6] denoted by $\Gamma^c(A, B)$. Its strategic representation is given by $\mathcal{G}^c(A, B) = (X_1^c, X_2^c, H_1^c, H_2^c)$, where for each player i a strategy is a pair $(\alpha_i, f_i) \in X_i^c$ with α_i a collection of transfer contracts and f_i a map which assigns an action $f_i(\bar{\alpha}) \in X_i$ to every pair $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2)$ of contract proposals. The payoff function is given by

$$H_i^c(\alpha, f) = \begin{cases} y_i & \text{if } \alpha_1 = \alpha_2 \text{ and } (f(\alpha), y) \in \alpha_i, \\ H_i(f(\alpha)) & \text{otherwise.} \end{cases}$$

Let us now consider the three examples discussed in Section 2. For the prisoners' dilemma of Example 2.1, the combination of strategies in which both players propose a set of contracts such that the cells (T, R) and (B, L) are replaced by $(0, 0)$ constitutes, in combination with player 1 (2) playing T (L) if both players choose these contracts and B (R) otherwise, a Nash equilibrium of the game $\Gamma^c(A, B)$. Formally, this equilibrium strategy profile $(\hat{\alpha}, \hat{f})$ is given by

$$\begin{aligned} \hat{\alpha}_1 = \hat{\alpha}_2 =& \{((T, R), (0, 0)), ((B, L), (0, 0))\}, \\ \hat{f}_1(\alpha) =& \begin{cases} T & \text{if } \alpha = \hat{\alpha}, \\ B & \text{otherwise,} \end{cases} \\ \hat{f}_2(\alpha) =& \begin{cases} L & \text{if } \alpha = \hat{\alpha}, \\ R & \text{otherwise.} \end{cases} \end{aligned}$$

It is clear that unilaterally deviating will not lead to a higher payoff: player i's best response is to play according to $\hat{f}_i$ in the second stage in case both players have played according to $\hat{\alpha}$ in the first stage, and in case player i has deviated in the first stage. In the latter situation the second stage consists of the original prisoners' dilemma game.

For the matching pennies in Example 2.2, we established that for instance (T, L), supported by $((0, 2), \{1, 2\})$ is a transfer equilibrium. However, $\Gamma^c(A, B)$ does not have a Nash equilibrium with associated payoff vector $(0, 2)$, as this outcome is only reachable if both players agree on a set of contracts. In that case, however, player 1 will deviate from (α, f) by choosing $\bar{\alpha}_1 = \emptyset$ in

[6] We model extensive form games as in Kreps and Wilson (1982).

combination with a best response to $f_2(\bar{\alpha}_1, \alpha_2)$, leading to a payoff equal to 2.

As a matter of fact, with a similar reasoning one can show that for matching pennies the contract game has no Nash equilibria at all. This is not a consequence of the game being constant-sum (as can be seen by replacing the (T, L) payoffs by (3,0), in which case the same arguments hold), but relates to the minimax payoffs of both players. In Section 4 we elaborate on this point.

The game of Example 2.3 does not have any transfer equilibria. The corresponding contract game $\Gamma^c(A, B)$, however, does possess a Nash equilibrium. Consider for instance the strategy combination in which both players propose a set of contracts such that all cells except (T, L) are replaced by $(0, 0)$. Furthermore, player 1 chooses T if this set of contracts is executed and B otherwise, and player 2 plays L regardless of the contract choice. Then unilaterally deviating will not lead to a higher payoff and hence, this combination of strategies is a Nash equilibrium in the contract game.

4 Folk theorems

In this section we analyse the equilibrium payoffs of a contract game. We provide necessary and sufficient conditions for a payoff vector to be the result of some Nash equilibrium of this two-stage game. Moreover, we give a similar result for an equilibrium refinement that appears natural in this context, called virtual subgame perfect equilibrium.

Given a two-player finite game (A, B), the *minimax payoff vector* $v \in \mathbb{R}$ is defined by

$$v_1 = \min_{s \in N} \max_{r \in M} a_{rs} \quad \text{and} \quad v_2 = \min_{r \in M} \max_{s \in N} b_{rs}.$$

Theorem 4.1 *Let (A, B) be a two-player finite game with minimax payoff vector v. For every $z \geq v$ such that $z_1 + z_2 \leq a_{rs} + b_{rs}$ for some $(r, s) \in M \times N$ there exists a Nash equilibrium of the game $\mathcal{G}^c(A, B)$ with corresponding payoff vector z.*

Proof: Let $z \geq v$ and $(\bar{r}, \bar{s}) \in M \times N$ be such that $z_1 + z_2 \leq a_{\bar{r}\bar{s}} + b_{\bar{r}\bar{s}}$. Let $(\tilde{r}, \tilde{s}) \in M \times N$ be such that

$$\max_{s \in N} b_{\tilde{r}s} = v_2 \quad \text{and} \quad \max_{r \in M} a_{r\tilde{s}} = v_1.$$

We construct a Nash equilibrium $(\hat{\alpha}, \hat{f})$ of the game $\mathcal{G}^c(A, B)$ as follows. For each player $i \in \{1, 2\}$, the set of transfer contracts is given by

$$\hat{\alpha}_i = \left\{ ((\bar{r}, \bar{s}), z) \right\} \cup \left\{ ((r, s), p) \,\middle|\, (r, s) \in (M \times N) \setminus \{(\bar{r}, \bar{s})\} \right\},$$

where $p \in \mathbb{R}^2$ is such that $p < v$ and $p_1 + p_2 \leq a_{rs} + b_{rs}$ for all $(r, s) \in (M \times N) \setminus \{(\bar{r}, \bar{s})\}$. The strategies in the second stage are given for all α by

$$\hat{f}_1(\alpha) = \begin{cases} \bar{r} & \text{if } \alpha = \hat{\alpha}, \\ \tilde{r} & \text{if } \alpha \neq \hat{\alpha}, \end{cases}$$

$$\hat{f}_2(\alpha) = \begin{cases} \bar{s} & \text{if } \alpha = \hat{\alpha}, \\ \tilde{s} & \text{if } \alpha \neq \hat{\alpha}. \end{cases}$$

Clearly, $H(\hat{\alpha}, \hat{f}) = z$. If player 1 chooses (α_1, f_1), then his payoff equals

$$H_1((\alpha_1, \hat{\alpha}_2), (f_1, \hat{f}_2)) = \begin{cases} z_1 & \text{if } \alpha_1 = \hat{\alpha}_1,\ f_1(\alpha_1, \hat{\alpha}_2) = \bar{r}, \\ p_1 & \text{if } \alpha_1 = \hat{\alpha}_1,\ f_1(\alpha_1, \hat{\alpha}_2) \neq \bar{r}, \\ H_1(f_1(\alpha_1, \hat{\alpha}_2), \tilde{s}) & \text{if } \alpha_1 \neq \hat{\alpha}_1. \end{cases}$$

Given the choice of p_1 and $\tilde{s}$,

$$H_1((\alpha_1, \hat{\alpha}_2), (f_1, \hat{f}_2)) \leq z_1 = H_1(\hat{\alpha}, \hat{f}).$$

Similarly, player 2 has no incentive to deviate and $(\hat{\alpha}, \hat{f})$ is a Nash equilibrium of $\mathcal{G}^c(A, B)$. □

Note that the condition on the payoff vector z in Theorem 4.1 is not only sufficient, but also necessary. If $z_i < v_i$ for some player $i \in \{1, 2\}$, then z is not an equilibrium payoff in the contract game, since player i will deviate by proposing no contract and playing his minimax strategy in the second stage.

Theorem 4.1 states that every feasible payoff vector larger than the minimax payoff vector of (A, B) is supported as the payoff of some Nash equilibrium of the contract game $\Gamma^c(A, B)$. However, the equilibrium strategy profile constructed in the proof may prescribe unreasonable strategy choices in subgames off the equilibrium path.

Consider for instance the game of Example 2.3, and the Nash equilibrium $(\hat{\alpha}, \hat{f})$ of the corresponding contract game $\Gamma^c(A, B)$ presented at the end of Section 3. In this game, $v = (6, 5)$ and $(\hat{\alpha}, \hat{f})$ is one of the Nash equilibria constructed in the proof of Theorem 4.1 with $(\bar{r}, \bar{s}) = (T, L)$, $z = (6, 6)$, $p = (0, 0)$, $\tilde{r} = B$ and $\tilde{s} = L$. The problem with this Nash equilibrium is that after any unilateral deviation from the proposed contract set the players end up in a subgame in which the original game (A, B) is played in the second stage, and in that game B is not a best response to L.

In order to deal with this shortcoming, one might consider subgame perfect equilibria of $\Gamma^c(A, B)$ (cf. Selten (1965)). A Nash equilibrium is called subgame perfect if it prescribes a Nash equilibrium in every subgame.

However, the set of subgame perfect equilibria in the contract game is always empty (if $m, n \geq 2$). Consider the subgame starting at the node where both players have proposed the same collection of contracts in such a way that the modified game in the second stage possesses no Nash equilibrium (notice that this can easily be done). Clearly, any proposed strategy in $\Gamma^c(A, B)$ does not prescribe a Nash equilibrium in this subgame and hence, no subgame perfect equilibrium exists.

The problem with the concept of subgame perfection in this particular model is that the game has too many subgames, some of which seem not particularly relevant. To tackle this problem, García-Jurado and Gonzalez-Díaz (2006) introduced the concept of *virtually subgame perfect equilibrium*. For a strategy profile σ in an extensive form game Γ to be a virtually subgame perfect equilibrium, it must prescribe a Nash equilibrium in the σ-relevant subgames of Γ. A subgame of Γ is called σ-relevant if it is Γ itself or if it starts at a node that can be reached from a σ-relevant subgame by at most one unilateral deviation from σ.

Let us once more consider the contract game corresponding to the game of Example 2.3. In the Nash equilibrium presented before, both players propose a set of contracts in which all cells except (T, L) are replaced by $(0, 0)$. Then given these contract choices, all σ-relevant subgames correspond to the second stage play of either the game in which all these contracts are executed, or the original game (A, B). This is due to the fact that if only one player deviates from his contract proposal the sets of proposed contracts do not match, in which case the original game (A, B) is played in the second stage. In order to end up in a different game in the second stage, both players have to deviate from the equilibrium strategy profile, which means that such a subgame is not σ-relevant.

Hence, a particular strategy profile can only be a virtually subgame perfect equilibrium if it results in a Nash equilibrium in the original game (A, B) for each subgame in which the players are called to play this game. Such a strategy profile obviously does not exist in the game of Example 2.3 as the subgames in which (A, B) is played in the second stage do not possess a Nash equilibrium.

Next, consider the prisoners' dilemma in Example 2.1 and the equilibrium strategy profile proposed in Section 3 for the corresponding contract game. Then we see that this strategy profile leads to a Nash equilibrium in all subgames in which (A, B) is played. Furthermore, it also constitutes a Nash equilibrium in the subgame in which the proposed contract set comes into effect. Therefore, this strategy profile is a virtually subgame perfect equilibrium.

These two examples indicate that there is a strong relation between the existence of Nash equilibria in the game (A, B) on the one hand and the existence of virtually subgame perfect equilibria in the contract game $\Gamma^c(A, B)$ on the other. The next theorem formalises this result.

Theorem 4.2 *Let (A,B) be a two-player finite game. For every $z \in \mathbb{R}^2$ such that for every $i \in \{1, 2\}$ there exists a Nash equilibrium (r^i, s^i) of (A, B) with $z_i \geq H_i(r^i, s^i)$ and such that $z_1 + z_2 \leq a_{rs} + b_{rs}$ for some $(r, s) \in M \times N$, there exists a virtually subgame perfect equilibrium of $\Gamma^c(A, B)$ with corresponding payoff vector z.*

Proof: Let z, (r^1, s^1) and (r^2, s^2) be as stated in the theorem and let $(\bar{r}, \bar{s}) \in M \times N$ be such that $z_1 + z_2 \leq H_1(\bar{r}, \bar{s}) + H_2(\bar{r}, \bar{s})$. Define the strategy profile $(\hat{\alpha}, \hat{f})$ as follows. For each $i \in \{1,2\}$,

$$\alpha_i = \{((\bar{r},\bar{s}),z)\} \cup \left\{ ((r,s),p) \;\middle|\; (r,s) \in (M \times N) \setminus \{(\bar{r},\bar{s})\} \right\},$$

where $p \in \mathbb{R}^2$ is such that $p < v$ (the minimax payoff vector) and $p_1 + p_2 \leq a_{rs} + b_{rs}$ for all $(r,s) \in (M \times N) \backslash \{(\bar{r},\bar{s})\}$. The strategies in the second stage are given for all α by

$$\hat{f}(\alpha) = \begin{cases} (\bar{r},\bar{s}) & \text{if } \alpha = \hat{\alpha}, \\ (r^1,s^1) & \text{if } \alpha_1 \neq \hat{\alpha}_1, \alpha_2 = \hat{\alpha}_2, \\ (r^2,s^2) & \text{if } \alpha_1 = \hat{\alpha}_1, \alpha_2 \neq \hat{\alpha}_2, \\ (r^*,s^*) & \text{otherwise,} \end{cases}$$

where (r^*,s^*) is an arbitrary strategy profile of (A,B).

Obviously, $H(\hat{\alpha}, \hat{f}) = z$. We check that $(\hat{\alpha}, \hat{f})$ is a virtually subgame perfect equilibrium of $\Gamma^c(A,B)$. First, in a similar way as in the proof of Theorem 4.1, one can show that $(\hat{\alpha}, \hat{f})$ is a Nash equilibrium of $\mathcal{G}^c(A,B)$.

The nodes which define a subgame (apart from the root) are the nodes corresponding to each profile of transfer contract collections (α_1, α_2). Of these, only the profiles reachable from unilateral deviations in the first stage, $(\hat{\alpha}_1, \alpha_2)$ and $(\alpha_1, \hat{\alpha}_2)$ give rise to $(\hat{\alpha}, \hat{f})$-relevant subgames.

Consider the subgame in which player 1 has chosen $\alpha_1 \neq \hat{\alpha}_1$ in the first stage. In this subgame, $\hat{f}$ prescribes (r^1, s^1), which is a Nash equilibrium in this subgame, because no contract is enforced. Similarly, $\hat{f}$ prescribes a Nash equilibrium in every $(\hat{\alpha}, \hat{f})$-relevant subgame in which player 2 has deviated. Hence, $(\hat{\alpha}, \hat{f})$ is a virtually subgame perfect equilibrium. □

Again, the conditions on the payoff vector z are necessary. For a strategy profile of $\Gamma^c(A,B)$ to be virtually subgame perfect, it has to prescribe a Nash equilibrium in the second stage in which the original game (A,B) is played, since this is always a relevant subgame. If, say, $H_1(r,s) > z_1$ for each Nash equilibrium (r,s) of (A,B), then player 1 has an incentive to deviate and propose no contract.

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